

Edge states for topological insulators in two dimensions and their Luttinger-like liquids

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Topological insulators in three spatial dimensions are known to possess a precise bulk/boundary correspondence, in that there is a one-to-one correspondence between the 5 classes characterized by bulk topological invariants and Dirac hamiltonians on the boundary with symmetry protected zero modes. This holographic characterization of topological insulators is studied in two dimensions. Dirac hamiltonians on the one dimensional edge are classified according to the discrete symmetries of time-reversal, particle-hole, and chirality, extending a previous classification in two dimensions. We find 17 inequivalent classes, of which 11 have protected zero modes. Although bulk topological invariants are thus far known for only 5 of these classes, we conjecture that the additional 6 describe edge states of new classes of topological insulators. The effects of interactions in two dimensions are also studied. We show that all interactions that preserve the symmetry are exactly marginal, i.e. preserve the gaplessness. This leads to a description of the distinct variations of Luttinger liquids that can be realized on the edge.

I. INTRODUCTION

Topological insulators are characterized by bulk wave-functions in d spatial dimensions with special topological properties characterized by certain topological invariants, such as the Chern number¹⁻⁸. These physical systems possess a kind of holography, or bulk/boundary correspondence, in that they necessarily have protected gapless excitations on the $\bar{d} = d - 1$ dimensional surface. These surface modes are typically described by Dirac hamiltonians. For example in the integer quantum Hall effect (QHE) in $d = 2$, the Chern number is the same integer as in the quantized Hall conductivity, and the edge states are chiral Dirac fermions.

Schnyder *et al.*⁹, Ryu *et al.*¹⁰ and Kitaev¹¹ classified topological insulators in any dimension according to the discrete symmetries of time reversal **T**, particle-hole symmetry **C** and chirality **P** and found 5 classes of topological insulators in any dimension. These classifications relied on generic properties in any dimension, namely the homotopy groups of replica sigma models for Anderson localization^{9,10}, or the 8-fold periodicity property of spinor representations of $so(n)$ based on their Clifford algebras,

which is a mild form of Bott-periodicity in K-theory¹¹.

The bulk/boundary correspondence was described explicitly in⁹ for $d = 3$ spatial dimensions: using the classification of $\bar{d} = 2$ dimensional Dirac hamiltonians in¹², it was found that precisely 5 of the 13 Dirac classes had protected surface states with the predicted discrete symmetries. In that analysis, it was crucial that the classification in¹² contained 3 additional classes beyond the 10 Altland-Zirnbauer (AZ) classes¹³, since it was precisely these additional classes that corresponded to some of the topological insulators. The reason that there are more classes of Dirac hamiltonians is that AZ classify finite dimensional hermitian matrices (hamiltonians) without assuming any Dirac structure.

In this paper we explore this ‘holographic classification’ of topological insulators (TI’s) and topological superconductors (TS’s) in $d=2$ spatial dimensions, in order to ascertain whether it works out as nicely as for $d=3$. The general d dimensional case will be presented elsewhere¹⁴. It is not obvious from the beginning that this holographic approach should reproduce precisely the classifications based on topological invariants. For instance, Anderson

localization properties are generally different in $d < 2$ versus $d > 2$. Furthermore, we assume that the surface states can be realized as Dirac fermions, which isn't necessarily the case.

This study requires a classification of Dirac hamiltonians in $\overline{d} = 1$, which is carried out for the first time below. We identify 17 unitarily-inequivalent classes. Since the classifications in^{9–11} were based on *generic* properties in any dimension, it is possible that there exist more classes of topological insulators in $d = 2$ due to this richer structure specific to $\overline{d} = 1$. Indeed, based on our classification, we find 11 classes of Dirac hamiltonians with protected zero modes on the 1 dimensional edge. In addition to the previously predicted topological insulators in classes A, C, D, DIII, and AII, we find the classes AIII, BDI, two versions of CII, an additional version of DIII, and a \mathbb{Z}_2 version of D (the definition of these classes will be reviewed below; the notation goes back to Cartan). One interpretation is that, unlike in $d = 3$, for $d = 2$ there are classes of $\overline{d} = 1$ Dirac hamiltonians that are protected for reasons other than the existence of a topological invariant for the $d = 2$ band structure. On the other hand, our new classes could in principle be characterized by some as yet unknown bulk topological invariants. Although this distinction needs to be kept in mind, henceforth, for simplicity, we will refer to all classes with protected zero modes on the boundary as TI's.

For the QHE, bulk interactions lead to the fractional QHE, and the effect of these interactions is that the edge states become Luttinger liquids¹⁵. This is unique to $d = 2$ since only in this dimension are quartic interactions on the boundary marginal, which is not unrelated to the fact that anyons only exist in 2 dimensions. Thus a criterion for the possible effects of bulk interactions is the existence of *exactly* marginal perturbations of the free boundary Dirac hamiltonian that are consistent with the discrete symmetries, since an exactly marginal perturbation deforms the theory but keeps it gapless. This leads us to also classify quartic, exactly marginal perturbations that are consistent with the discrete symmetries. In addition to the or-

dinary, chiral and helical Luttinger liquids, we find the possibility of 3 additional varieties in the classes DIII and CII.

The sections below cover the following. In section II we review the definitions of the 10 AZ classes. Section III reviews the holographic classification of TI in $d = 3$. One-dimensional Dirac hamiltonians are classified in section IV. This classification is completely general, and could have applications in other areas, such as disordered systems. In section V, we identify the Dirac theories with protected zero modes, and section VI describes their consequent Luttinger liquids.

II. DISCRETE SYMMETRIES

The 10 Altland-Zirnbauer (AZ) classes of random hamiltonians arise when one considers time reversal symmetry (**T**), particle-hole symmetry (**C**), and parity or chirality (**P**). These discrete symmetries are defined to act as follows on a first-quantized hamiltonian \mathcal{H} :

$$\begin{aligned} \mathbf{T} : & \quad T\mathcal{H}^*T^\dagger = \mathcal{H} \\ \mathbf{C} : & \quad C\mathcal{H}^TC^\dagger = -\mathcal{H} \\ \mathbf{P} : & \quad P\mathcal{H}P^\dagger = -\mathcal{H} \end{aligned} \quad (1)$$

with $TT^\dagger = CC^\dagger = PP^\dagger = \mathbf{1}$. We consider two hamiltonians $\mathcal{H}, \mathcal{H}'$ related by a unitary transformation $\mathcal{H}' = U\mathcal{H}U^\dagger$ to be in the same class, since they have the same eigenvalues. For C and T , this translates to $C \rightarrow C' = UCUT$ and $T \rightarrow T' = UTU^T$. For P , it amounts to $P \rightarrow P' = UPU^\dagger$. It is thus important to identify these unitary equivalences in order not to over-count classes.

For hermitian hamiltonians, $\mathcal{H}^T = \mathcal{H}^*$, thus, up to a sign, **C** and **T** symmetries are the same. We focus then on these symmetries involving the transpose: $T\mathcal{H}^TT^\dagger = \mathcal{H}$ and $C\mathcal{H}^TC^\dagger = -\mathcal{H}$. Taking the transpose of this relation, one finds there are two consistent possibilities: $T^T = \pm T, C^T = \pm C$, which are unitarily-invariant relations. It turns out that unitary transformations allow us to choose T, C to be real; unitarity of T, C then implies $C^2 =$

| AZ-classes | T^2 | C^2 | P^2 |
|------------|-------------|-------------|-------------|
| A | \emptyset | \emptyset | \emptyset |
| AIII | \emptyset | \emptyset | 1 |
| AII | -1 | \emptyset | \emptyset |
| AI | +1 | \emptyset | \emptyset |
| C | \emptyset | -1 | \emptyset |
| D | \emptyset | +1 | \emptyset |
| BDI | +1 | +1 | 1 |
| DIII | -1 | +1 | 1 |
| CII | -1 | -1 | 1 |
| CI | +1 | -1 | 1 |

TABLE I. The 10 Altland-Zirnbauer (AZ) hamiltonian classes. \emptyset denotes the absence of respective symmetry.

$\pm 1, T^2 = \pm 1$. The various classes are thus distinguished by $T^2 = \pm 1, \emptyset$ and $C^2 = \pm 1, \emptyset$, where \emptyset indicates that the hamiltonian does not have the symmetry, and the sign is equivalent to the sign in the relation between T, C and their transpose. One obtains $9 = 3 \times 3$ classes just by considering the 3 cases for **T** and **C**. If the hamiltonian has both **T** and **C** symmetry, then it automatically has a **P** symmetry, with $P = TC^\dagger$ up to a phase. If there is neither **T** nor **C** symmetry, then there are two choices $P = \emptyset, 1$, and this gives the additional class AIII, leading to a total of 10. Their properties are shown in Table I. We also mention that one normally requires $P^2 = 1$. Below, we will require **T** and **C** to commute, thus $P^2 = T^2 C^{\dagger 2} = \pm 1$. However one has the freedom $P \rightarrow iP$ to restore $P^2 = 1$. In the sequel, in the cases with both **T, C** symmetry, we simply define $P = TC^\dagger$, up to a phase.

III. REVIEW OF THE $\bar{d} = 2$ DIMENSIONAL CASE

The connection between the bulk topological properties and the existence of protected zero modes on the boundary was first pointed out for $d = 3$ by Schnyder et. al.⁹. This relied

on the classification of $\bar{d} = 2$ dimensional Dirac hamiltonians found by two of us¹². In this section we review this holographic classification of $d = 3$ TI's since this illustrates what we are attempting to accomplish in $d = 2$.

If one requires a Dirac structure of the hamiltonian, then the AZ classification can be more refined. The most general hamiltonian in $\bar{d} = 2$ dimensions is of the form:

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\partial_{\bar{z}} + A_{\bar{z}} \\ -i\partial_z + A_z & V_+ - V_- \end{pmatrix} \quad (2)$$

where $\partial_z = \partial_x - i\partial_y, \partial_{\bar{z}} = \partial_x + i\partial_y$ with x, y the spatial coordinates and $V_{\pm}, A_{z, \bar{z}}$ are matrices. The above \mathcal{H} is just a relabeling of $\mathcal{H} = -i\sigma_x \partial_x - i\sigma_y \partial_y + \vec{\sigma} \cdot \vec{V} + V_0$, i.e. the block structure comes from the Pauli matrices σ .

One then finds the most general form of the T, C, P matrices that preserve the Dirac structure. Thirteen inequivalent classes were found¹². In particular, there exist two inequivalent versions of the chiral classes AIII, DIII, and CI, simply because the discrete symmetries can take different forms. It was shown in⁹ that precisely 5 of the 13 classes corresponded to the surface states of TI's, with discrete symmetries consistent with the predictions from bulk topology. As argued there, the criterion for a TI is that V_- has a zero mode, i.e. $\det V_- = 0$. This led to the following identification of TI's, where the nomenclature of¹² is given in parentheses. As far as the bulk properties, are two types of topological invariants, \mathbb{Z} and \mathbb{Z}_2 , which are also indicated. In the holographic approach, \mathbb{Z} versus \mathbb{Z}_2 corresponds to the two ways of obtaining a zero mode, namely $V_- = 0$ or $\det V_- = -\det V_-$ for V_- odd dimensional (see section VA).

- **AIII (1) , DIII (5) , CI (6)** . These are the three classes that are doubled in comparison with AZ. For one of the two in each these classes, the discrete symmetry forces $V_- = 0$. These are all of type \mathbb{Z} .
- **AII (3₊)**. Here the discrete symmetries require $V_-^T = -V_-$, which implies that if V_- is odd-dimensional, $\det V_- = 0$. Type \mathbb{Z}_2 .

- **CII** (9₋). In this case, the discrete symmetries constrain $V_- = \begin{pmatrix} 0 & v_- \\ w_- & 0 \end{pmatrix}$ with $v_-^T = -v_-$, $w_-^T = -w_-$. Thus if v_- , w_- are odd-dimensional, then up to a sign, $\det V_- = \det v_- \det w_- = 0$. Type \mathbb{Z}_2 .

IV. THE $\bar{d} = 1$ DIMENSIONAL CLASSIFICATION OF DIRAC HAMILTONIANS

In this section, we present the complete classification of $\bar{d} = 1$ dimensional Dirac hamiltonians. Although the identification of TI's and TS's will be the subject of the next section, it is useful to motivate what follows by discussing chiral Dirac Hamiltonians with only right moving or left moving fermions¹⁶. Since a mass term necessarily couples left and right movers (see section V), these classes have a protected zero mode for somewhat trivial reasons. Such Hamiltonians cannot be realized on a 1d lattice and they necessarily break **T** and **P**. However they can appear as a $\bar{d} = 1$ -edge state of a 2d TI or TS in classes A, C, and D which break both **T** and **P**. An example of class A is the quantum Hall effect. Depending on the number of filled Landau levels there are \mathbb{Z} number of edge states¹. An example of class C is the spin quantum Hall effect in a singlet time-reversal breaking superconductor. The spin quantum Hall conductivity will be proportional to the Cooper pair angular momentum, hence this is a \mathbb{Z} TS. Although there is no known experimental realization, $d_{x^2-y^2} + id_{xy}$ superconductor (SC) was extensively discussed theoretically^{17,18}. A realization of class D would be the thermal Hall effect of a time-reversal breaking superfluid of spinless (fully spin polarized) fermions. The $\nu = 5/2$ quantum Hall state could be a $p_x + ip_y$ paired superfluid of composite fermions¹⁹.

All non-“chiral” non-interacting 1d Dirac hamiltonians with equal number of right-movers and left-movers can be written as $\mathcal{H} = -i\sigma_x \partial_x + \vec{\sigma} \cdot \vec{A} + V_+$, where $\vec{\sigma}$ are the Pauli matrices acting on a space of right/left-movers $|\sigma_x = \pm\rangle$.

Redefining $A_z = V_-$, these hamiltonians can be expressed as

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\partial_x + A \\ -i\partial_x + A^\dagger & V_+ - V_- \end{pmatrix}. \quad (3)$$

The potentials V_\pm are hermitian matrices and $A = A_x + iA_y$ where $A_{x,y}$ are also hermitian matrices in general. The dimension of V_\pm and A is the number of edge mode species for each chirality. When V_\pm and A are even dimensional we use $\vec{\tau}$ to denote a set of Pauli matrices acting on the even dimensional flavor space. **1** will denote the identity in either the σ or τ space. Note that $\vec{\sigma}$ and $\vec{\tau}$ have distinct physical meaning: $\vec{\sigma}$ acts on the space of “chirality” as we show explicitly in section VB, and it is responsible for the block structure of Eq.(3), whereas $\vec{\tau}$ acts on the space of flavors which could be spin or pseudo-spin. If there is spin-momentum locking (see section VB) $\vec{\sigma}$ will act on the spin space as well as on the space of “chirality”.

The Dirac derivative structure of \mathcal{H} constrains the form of T, C , and P in terms of $\vec{\sigma}$ and $\vec{\tau}$. Furthermore, we can specify the conditions V_\pm and A have to satisfy in order for \mathcal{H} to have discrete symmetries under specific T, C , or P . Hence the specific forms of symmetry transformations can be used to classify hamiltonians of form Eq.(3). Since, as described below, there are multiple sets of matrices T, C, P with the same T^2, C^2, P^2 , this scheme refines the AZ classification of Table I. Here we find even more classes of Dirac hamiltonians in $\bar{d} = 1$ than in $\bar{d} = 2$, and more classes with symmetry protected zero modes (see section V).

In the rest of this section, we first specify the forms of T, C and P symmetry that preserve the Dirac structure, and describe the resulting conditions on V_\pm and A in a fixed $\vec{\sigma}$ basis and arrive at 25 classes as summarized in Table II. We then check for unitary equivalences. The unitary transform is

$$\mathcal{H} \rightarrow U_\theta \mathcal{H} U_\theta^\dagger \quad (4)$$

with U_θ a rotation about the x -axis in σ -space

by an angle θ :

$$U_\theta = u \cdot e^{i\theta\sigma_x/2} = u \cdot (\mathbf{1} \cos(\theta/2) + i\sigma_x \sin(\theta/2)) \quad (5)$$

where u is unitary and commutes with σ_x . We find 17 unitarily-inequivalent classes, each forming a row separated by a horizontal line in Table II.

Consider first the **T** symmetry. In order to preserve the derivative structure of the hamiltonian Eq.(3), using $(-i\partial_x)^T = i\partial_x$, one finds that T must anti-commute with σ_x . Since T is (anti)-symmetric and unitary, it is then either proportional to σ_z or $i\sigma_y$. This leads to 2 ways of implementing of **T**-symmetry transformations: using either

$$T = \eta_t \otimes i\sigma_y = \begin{pmatrix} 0 & \eta_t \\ -\eta_t & 0 \end{pmatrix} \quad (6)$$

$$\tilde{T} = \tilde{\eta}_t \otimes \sigma_z = \begin{pmatrix} \tilde{\eta}_t & 0 \\ 0 & -\tilde{\eta}_t \end{pmatrix}, \quad (7)$$

where η_t or $\tilde{\eta}_t$ are unitary matrices in general. Then, for a hamiltonian of form Eq.(3) to have **T** symmetry the potentials have to satisfy either

$$\eta_t V_\pm^T = \pm V_\pm \eta_t, \quad \eta_t A^T = -A \eta_t \quad (8)$$

or

$$\tilde{\eta}_t V_\pm^T = V_\pm \tilde{\eta}_t, \quad \tilde{\eta}_t A^* = -A \tilde{\eta}_t \quad (9)$$

Now the condition $T^T = \pm T$ ($T^2 = \pm 1$) which distinguishes AI from AII for instance, implies either $\eta_t^T = \pm \eta_t$ or $\tilde{\eta}_t^T = \pm \tilde{\eta}_t$. Hence all AZ classes with **T**-symmetry are further refined depending on whether T (Eq.(6)) or \tilde{T} (Eq.(7)) is used to implement **T**. This distinction has a physical significance: the use of $T \propto i\sigma_y$ leads to spin-momentum locking (see section VB).

Finally we can choose representations of η_t in terms of $\vec{\tau}$ up to the unitary transformations: $\eta_t = \mathbf{1}$ if $\eta_t^T = \eta_t$, and $\eta_t = i\tau_y$ if $\eta_t^T = -\eta_t$ ²⁰. We can do the same for $\tilde{\eta}_t$. The unitary transform $T \rightarrow UTU^T$ corresponds to $\eta \rightarrow u\eta u^T$ with u unitary, for all η 's. The unitary transformation affects the choice of $\mathbf{1}$ v.s. τ_x for

η_t 's. However the unitary transform cannot affect the distinction between T and \tilde{T} . In particular when **T** is the only available discrete symmetry, $T^2, \tilde{T}^2 = \pm 1$ completely classifies $\bar{d} = 1$ Dirac Hamiltonians into AI₍₁₎, AI₍₂₎ and AII₍₁₎, AII₍₂₎. See Table II.

We can specify C , following steps analogous to those for specifying T . As C must commute with σ_x for Dirac hamiltonian Eq.(3), it is in the linear span of $\mathbf{1}$ and σ_x . Hence there are two possibilities:

$$C = \eta_c \otimes \sigma_x, \quad \eta_c V_\pm^T = \mp V_\pm \eta_c, \quad \eta_c A^T = -A \eta_c \quad (10)$$

$$\tilde{C} = \tilde{\eta}_c \otimes \mathbf{1}, \quad \tilde{\eta}_c V_\pm^T = -V_\pm \tilde{\eta}_c, \quad \tilde{\eta}_c A^* = -A \tilde{\eta}_c$$

with η_c and $\tilde{\eta}_c$ unitary. The condition $C^T = \pm C$ that distinguishes AZ class C from D for instance, implies that $\eta_c^T = \pm \eta_c$ or $\tilde{\eta}_c^T = \pm \tilde{\eta}_c$. One can again represent up to unitary transformations $\eta_c = \mathbf{1}$ if $\eta_c^T = \eta_c$, and $\eta_c = i\tau_y$ if $\eta_c^T = -\eta_c$. This again refines the AZ classes with **C** symmetry. However unlike T and \tilde{T} which are unitarily-inequivalent, C and \tilde{C} are unitarily-equivalent for non-zero A_y (see the end of this section). We denote such unitarily-equivalent refinements using primed notation within the same row in Table II. In particular, this completes our classification of $\bar{d} = 1$ Dirac hamiltonians with only **C** symmetry into C, C', D, D'.

Consider now **P** symmetry. P must anti-commute with σ_x for the Dirac hamiltonian Eq.(3), so P is in the linear span of σ_y and σ_z . For P unitary, this implies that $P = \eta_p \cdot (\cos b \sigma_y + \sin b \sigma_z)$ for some real b . All these choices are unitarily-equivalent by rotations around the x -axis in the sigma space. However, in order to accommodate $P = TC^\dagger$ in all cases, we define two unitarily-equivalent types:

$$P = \eta_p \otimes \sigma_z \quad \eta_p V_\pm = -V_\pm \eta_p, \quad \eta_p A = A \eta_p \quad (11)$$

$$\tilde{P} = \tilde{\eta}_p \otimes i\sigma_y \quad \tilde{\eta}_p V_\pm = \mp V_\pm \tilde{\eta}_p, \quad \tilde{\eta}_p A^\dagger = A \tilde{\eta}_p$$

where η_p and $\tilde{\eta}_p$ are unitary. The unitary freedom reduces to $\eta_p \rightarrow u\eta_p u^\dagger$ and the same for

| 1d-classes | T | C | P | V_{\pm} | A | zero-mode |
|----------------------|--------------------------------|---------------------------------|--------------------------------|---|---|----------------------------|
| A | \emptyset | \emptyset | \emptyset | $V_{\pm}^{\dagger} = V_{\pm}$ | | \mathbb{Z} |
| AIII ₍₁₎ | \emptyset | \emptyset | $\mathbf{1} \otimes \sigma_z$ | $V_{\pm} = 0$ | | \mathbb{Z} |
| AIII' ₍₁₎ | \emptyset | \emptyset | $\mathbf{1} \otimes i\sigma_y$ | $V_{+} = 0$ | | |
| AIII ₍₂₎ | \emptyset | \emptyset | $\tau_z \otimes \sigma_z$ | $\tau_z V_{\pm} = -V_{\pm} \tau_z$ | $\tau_z A = A \tau_z$ | |
| AIII' ₍₂₎ | \emptyset | \emptyset | $\tau_z \otimes i\sigma_y$ | $\tau_z V_{\pm} = \mp V_{\pm} \tau_z$ | | |
| AII ₍₁₎ | $\mathbf{1} \otimes i\sigma_y$ | \emptyset | \emptyset | $V_{\pm} = \pm V_{\pm}^T$ | $A^T = -A$ | \mathbb{Z}_2 |
| AII ₍₂₎ | $i\tau_y \otimes \sigma_z$ | \emptyset | \emptyset | $\tau_y V_{\pm}^T = V_{\pm} \tau_y$ | $\tau_y A^* = -A \tau_y$ | |
| AI ₍₁₎ | $i\tau_y \otimes i\sigma_y$ | \emptyset | \emptyset | $\tau_y V_{\pm}^T = \pm V_{\pm} \tau_y$ | $\tau_y A^T = -A \tau_y$ | |
| AI ₍₂₎ | $\mathbf{1} \otimes \sigma_z$ | \emptyset | \emptyset | $V_{\pm}^T = V_{\pm}$ | $A^* = -A$ | |
| C | \emptyset | $i\tau_y \otimes \mathbf{1}$ | \emptyset | $\tau_y V_{\pm}^T = -V_{\pm} \tau_y$ | $\tau_y A^* = -A \tau_y$ | \mathbb{Z} |
| C' | \emptyset | $i\tau_y \otimes \sigma_x$ | \emptyset | $\tau_y V_{\pm}^T = \mp V_{\pm} \tau_y$ | $\tau_y A^T = -A \tau_y$ | |
| D | \emptyset | $\mathbf{1} \otimes \mathbf{1}$ | \emptyset | $V_{\pm} = -V_{\pm}^T$ | $A^* = -A$ | \mathbb{Z}, \mathbb{Z}_2 |
| D' | \emptyset | $\mathbf{1} \otimes \sigma_x$ | \emptyset | $V_{\pm} = \mp V_{\pm}^T$ | $A^T = -A$ | |
| BDI ₍₁₎ | $i\tau_y \otimes i\sigma_y$ | $\mathbf{1} \otimes \mathbf{1}$ | $i\tau_y \otimes i\sigma_y$ | $V_{\pm} = -V_{\pm}^T = \mp \tau_y V_{\pm} \tau_y$ | $A = -A^* = -\tau_y A^T \tau_y$ | |
| BDI' ₍₁₎ | $i\tau_y \otimes i\sigma_y$ | $\tau_x \otimes \sigma_x$ | $\tau_z \otimes \sigma_z$ | $V_{\pm} = \pm \tau_y V_{\pm}^T \tau_y = \mp \tau_x V_{\pm}^T \tau_x$ | $\tau_{x,y} A^T = -A \tau_{x,y}$ | |
| BDI ₍₂₎ | $\mathbf{1} \otimes \sigma_z$ | $\mathbf{1} \otimes \mathbf{1}$ | $\mathbf{1} \otimes \sigma_z$ | $V_{\pm} = 0$ | $A^* = -A$ | \mathbb{Z} |
| DIII ₍₁₎ | $\mathbf{1} \otimes i\sigma_y$ | $\mathbf{1} \otimes \mathbf{1}$ | $\mathbf{1} \otimes i\sigma_y$ | $V_{+} = 0, V_{-}^T = -V_{-}$ | $A = -A^* = -A^T$ | \mathbb{Z}_2 |
| DIII ₍₂₎ | $i\tau_y \otimes \sigma_z$ | $\mathbf{1} \otimes \mathbf{1}$ | $i\tau_y \otimes \sigma_z$ | $V_{\pm} = -V_{\pm}^T = -\tau_y V_{\pm} \tau_y$ | $A = -A^* = -\tau_y A^T \tau_y$ | \mathbb{Z}_2 |
| DIII' ₍₂₎ | $i\tau_y \otimes \sigma_z$ | $\tau_x \otimes \sigma_x$ | $\tau_z \otimes i\sigma_y$ | $V_{\pm} = \tau_y V_{\pm}^T \tau_y = \mp \tau_x V_{\pm}^T \tau_x$ | $A = -\tau_y A^* \tau_y = -\tau_x A^T \tau_x$ | |
| CII ₍₁₎ | $\mathbf{1} \otimes i\sigma_y$ | $i\tau_y \otimes \mathbf{1}$ | $i\tau_y \otimes i\sigma_y$ | $V_{\pm} = \pm V_{\pm}^T = \mp \tau_y V_{\pm} \tau_y$ | $A = -A^T = -\tau_y A^* \tau_y$ | \mathbb{Z}_2 |
| CII' ₍₁₎ | $\tau_x \otimes i\sigma_y$ | $i\tau_y \otimes \sigma_x$ | $\tau_z \otimes \sigma_z$ | $V_{\pm} = \pm \tau_x V_{\pm}^T \tau_x = \mp \tau_y V_{\pm}^T \tau_y$ | $\tau_{x,y} A^T = -A \tau_{x,y}$ | |
| CII ₍₂₎ | $i\tau_y \otimes \sigma_z$ | $i\tau_y \otimes \mathbf{1}$ | $\mathbf{1} \otimes \sigma_z$ | $V_{\pm} = 0$ | $A = -\tau_y A^* \tau_y$ | \mathbb{Z} |
| CI ₍₁₎ | $i\tau_y \otimes i\sigma_y$ | $i\tau_y \otimes \mathbf{1}$ | $\mathbf{1} \otimes i\sigma_y$ | $V_{+} = 0, \tau_y V_{-}^T = -V_{-} \tau_y$ | $A = -\tau_y A^T \tau_y = -\tau_y A^* \tau_y$ | |
| CI ₍₂₎ | $\mathbf{1} \otimes \sigma_z$ | $i\tau_y \otimes \mathbf{1}$ | $i\tau_y \otimes \sigma_z$ | $V_{\pm} = V_{\pm}^T = -\tau_y V_{\pm} \tau_y$ | $A = -A^* = -\tau_y A^* \tau_y$ | |
| CI' ₍₂₎ | $\tau_x \otimes \sigma_z$ | $i\tau_y \otimes \sigma_x$ | $\tau_z \otimes i\sigma_y$ | $V_{\pm} = \tau_x V_{\pm}^T \tau_x = \mp \tau_y V_{\pm}^T \tau_y$ | $A = -\tau_x A^* \tau_x = -\tau_y A^T \tau_y$ | |

TABLE II. The properties of the 25 non-chiral $\bar{d} = 1$ Dirac classes. 17 unitarily-inequivalent classes separated from each other by a horizontal line. The first column lists the $\bar{d} = 1$ Dirac classes. Columns **T**, **C** and **P** show representations of symmetry transformations for each class. The columns V_{\pm} and A show symmetry constraints on the potentials. A blank cell denotes absence thereof. The symmetry constraints guarantee zero modes in some classes (see section V). The last column shows classes with symmetry protected zero modes and the type of zero modes.

$\tilde{\eta}$. Up to unitary transformations there are two choices: $\eta_p, \tilde{\eta}_p = 1$ or τ_z . This gives 4 AIII classes.

Finally for the classes with both **T**, **C** symmetries we require that **T** and **C** commute since their physical origins are assumed to be unrelated. This implies that the η 's should commute

or anti-commute:

$$\{\eta_t, \eta_c\} = \{\tilde{\eta}_t, \eta_c\} = 0 = [\tilde{\eta}_t, \tilde{\eta}_c] = [\eta_t, \tilde{\eta}_c] = 0. \quad (12)$$

While **T** and **C** each can be implemented using either $T = \eta_t \otimes i\sigma_y$ or $\tilde{T} = \tilde{\eta}_t \otimes \sigma_z$, and either $\tilde{C} = \tilde{\eta}_c \otimes \mathbf{1}$ or $C = \eta_c \otimes \sigma_x$ respectively, the requirement of Eq.(12) rules out (\tilde{T}, C) possibility. This yields three possibilities: (T, C) ,

(\tilde{T}, C) and (\tilde{T}, \tilde{C}) . While A_y is non-zero, (\tilde{T}, C) and (\tilde{T}, \tilde{C}) possibilities are gauge-equivalent.

Now the AZ classes BDI, CI, DIII, and CII refines into 12 classes; among these 8 are gauge inequivalent. We label the three subclasses associated with the BDI class by $\text{BDI}_{(1)}$, $\text{BDI}_{(2)}$, $\text{BDI}'_{(2)}$, and similarly for CI, DIII, and CII. Table II shows this classification with respective representations of \mathbf{T} , \mathbf{C} and \mathbf{P} . In some cases η_t or η_c had to be taken to be τ_x which is unitarily-equivalent to $\mathbf{1}$, in order to satisfy that the anti-commutator in (12) equal zero. When there are both \mathbf{T}, \mathbf{C} symmetries, then there is automatically a $P = TC^\dagger$ symmetry (up to a phase). Depending on the type of C, T , one finds the \mathbb{Z}_2 graded multiplication: $P = TC^\dagger, P = \tilde{T}\tilde{C}^\dagger, \tilde{P} = T\tilde{C}^\dagger, \tilde{P} = \tilde{T}C^\dagger$. This gives $\eta_p = \eta_t\eta_c^\dagger$ or $\tilde{\eta}_t\tilde{\eta}_c^\dagger$ and $\tilde{\eta}_p = \eta_t\tilde{\eta}_c^\dagger$ or $\tilde{\eta}_t\eta_c^\dagger$.

Let us finally return to the issue of unitary equivalence. The unitary transform of Eq. (4) preserves the Dirac structure for U_θ of Eq. (5). The two possibilities T and \tilde{T} for \mathbf{T} are unitarily-inequivalent, because unitary transformations preserve the relation $T^T = \pm T$, or equivalently, $U_\theta\sigma_{y,z}U_\theta^T = \sigma_{y,z}$. However C and \tilde{C} are unitarily-equivalent for non-zero A_y , since $U_{\pi/2}\sigma_xU_{\pi/2}^T = i$. In Table II, we listed all 25 classes separating 17 unitarily-inequivalent classes by horizontal lines. It is important to note however that all of the 25 classes should be viewed as inequivalent once U_θ is used to set $A_y = 0$ since C, \tilde{C} are inequivalent under the residual symmetry. (If $A_y = 0$, $A^* = A^T$.) We will take this route in the next section where we investigate the symmetry protection of zero modes.

V. “TOPOLOGICAL INSULATORS” IN TWO DIMENSIONS

We conjecture a ‘holographic’ classification of 2D TI-TS based on the classification of $\bar{d} = 1$ Dirac hamiltonians that are symmetry protected to be gapless, i.e. have a protected zero mode. We list such $\bar{d} = 1$ Dirac hamiltonian classes in Tables III and IV. For a *subset* of these

classes, there exists a $d = 2$ gapped hamiltonian in the same class and a known topological invariant which one can calculate from the ground state wave function which takes on \mathbb{Z} -values or \mathbb{Z}_2 -values^{9,10}; these are indicated in the columns denoted “topological invariant”. Surprisingly, for a class with a known bulk topological invariant, there is a correspondence between the values it can take and the number of gapless Dirac edge branches (dimension of the block matrices Eq(3) for the non-chiral case). Namely, classes with \mathbb{Z} -invariants are gapless for any number of Dirac edge branches; classes with \mathbb{Z}_2 -invariants are gapless only when there are odd-number of branches for each chirality. The main point of this paper is that there are additional classes with protected edge zero modes beyond the 5 predicted on the basis of the known topological invariants.

In the rest of this section we enumerate the classes of $\bar{d} = 1$ Dirac hamiltonians that have a protected zero mode as a consequence of the discrete symmetries. We discuss physical properties of these classes such as spin-momentum locking through a second quantized description. We then discuss possible examples of physical realizations.

A. First quantized description

First we discuss the chiral (only right or left moving) Dirac fermion classes we mentioned at the beginning of section IV. These are massless for a “trivial” reason since a mass term necessarily couples left to right. As \mathbf{T} and \mathbf{P} transform left to right movers (see below), hamiltonians with these symmetries cannot be chiral. On the other hand, AZ classes A, C, D have at most a \mathbf{C} symmetry and can be chiral. For chiral hamiltonians in classes A, C, D, any \mathbb{Z} number of branches will be gapless. See Table III for the summary.

Now consider non-chiral hamiltonians of the form Eq. (3) whose block diagonal structure implies that the second quantized theory has both right movers $\psi_R \equiv \langle x|\sigma_x = + \rangle$ and left movers $\psi_L \equiv \langle x|\sigma_x = - \rangle$ (see below). The hamiltonian

| $\bar{d} = 1$ classes | zero modes | topological invariant | examples |
|-----------------------|--------------|-----------------------|--|
| A | \mathbb{Z} | \mathbb{Z} | QH edge states |
| C | \mathbb{Z} | \mathbb{Z} | spin QH edge states in $d + id$ -wave SC ^{17,18} |
| D | \mathbb{Z} | \mathbb{Z} | thermal QH edge states in spinless chiral p -wave SC ¹⁷ |

TABLE III. $\bar{d} = 1$ chiral Dirac hamiltonian classes.

| $\bar{d} = 1$ classes | T | C | P | zero modes | top. inv. | locking | examples |
|----------------------------|---|------------------------------|--|----------------|----------------|---------|-------------------------------------|
| AIII ₍₁₎ | \emptyset | \emptyset | σ_z | \mathbb{Z} | | | |
| AII ₍₁₎ | $i\sigma_y$ | \emptyset | \emptyset | \mathbb{Z}_2 | \mathbb{Z}_2 | Y | HgTe/(Hg,Cd)Te |
| D | \emptyset | 1 | \emptyset | \mathbb{Z}_2 | | | |
| BDI ₍₂₎ | σ_z | 1 | σ_z | \mathbb{Z} | | | “strained graphene” |
| DIII ₍₁₎ | $i\sigma_y$ | 1 | $i\sigma_y$ | \mathbb{Z}_2 | \mathbb{Z}_2 | Y | $(p + ip) \times (p - ip)$ -wave SC |
| DIII ₍₂₎ | $i\tau_y \otimes \sigma_z$ | 1 | $i\tau_y \otimes \sigma_z$ | \mathbb{Z}_2 | \mathbb{Z}_2 | N | particle-hole symmetric KM model |
| CII ₍₁₎ | $1 \otimes i\sigma_y$ | $i\tau_y \otimes \mathbf{1}$ | $i\tau_y \otimes i\sigma_y$ | \mathbb{Z}_2 | | Y | doubled KM |
| CII ₍₂₎ | $i\tau_y \otimes \sigma_z$ | $i\tau_y \otimes \mathbf{1}$ | $1 \otimes \sigma_z$ | \mathbb{Z} | | N | |

TABLE IV. $\bar{d} = 1$ non-chiral Dirac hamiltonian classes with symmetry protected zero modes. The spin-momentum locking column is left blank when spins cannot be assigned because the time-reversal operator do not involve either $i\sigma_y$ or $i\tau_y$. New classes are shown in boldface (red online). The example in quotation marks is a *suggested* possible realization.

\mathcal{H} is gapless if it has a zero eigenvalue at $\mathbf{k} = 0$, i.e. $\det \mathcal{H}(\mathbf{k} = 0) = 0$. Below we simplify this into a condition on V_- .

The potential A_x can be removed by redefining the fields in the second quantized theory: $\psi_{L,R} \rightarrow e^{-i \int^x A_x(x) dx} \psi_{L,R}$ (see subsection V B). A constant V_+ is a chemical potential which shifts the overall energy levels. Hence we set this to zero. Now the condition for existence of a zero mode and hence a gapless spectrum is

$$\det \begin{pmatrix} V_- & iA_y \\ -iA_y & -V_- \end{pmatrix} = 0 \quad (13)$$

However Eq. (13) is difficult to use in general²¹. Hence we use the freedom of unitary transform U_θ to set $A_y = 0$. The criterion for a TI is now simply $\det V_- = 0$ for fixed $A_y = 0$.

Now we test if the conditions on V_- imposed by symmetry listed in Table II guarantee $\det V_- = 0$. As the choice of $A_y = 0$ makes C and \tilde{C} inequivalent we consider all 25 entries. Once we identify symmetry protected gapless

Dirac classes, we check for unitary equivalence among those by consulting the Table II. In Table IV we list unitarily inequivalent protected classes.

Two types of constraints on V_- protect a gapless spectrum. First, $V_- = 0$ guarantees $\det V_- = 0$ independent of the dimension of V_- nor the \mathbb{Z} - number of edge modes. Second, $V_-^T = -V_-$ implies $\det V_- = -\det V_-$ when V_- is *odd dimensional*, and hence $\det V_- = 0$. For 3d TI-TS the $\bar{d} = 2$ Dirac classes with $V_- = 0$ and those with $V_-^T = -V_-$ each corresponded respectively to TI-TS with \mathbb{Z} and \mathbb{Z}_2 bulk topological invariants⁹. By analogy with the 3d case, the possible 2d TI-TS where $V_- = 0$ should be \mathbb{Z} type, whereas those that rely on $V_-^T = -V_-$ with V_- odd-dimensional should be of \mathbb{Z}_2 type because of the even/odd aspect. We summarize the resulting 2d “TI-TS” classes in the table IV. Note that all TI-TS’s so-identified are unitarily inequivalent, as they must.

B. Second quantized description and spin-momentum locking

One can define a second-quantized hamiltonian:

$$H = \int dx \sum_{a,b} \psi_a^\dagger(x) \mathcal{H}_{ab} \psi_b(x) \quad (14)$$

from \mathcal{H} of Eq. (3). Now let \mathbf{T}, \mathbf{C} be time-reversal and particle-hole transformation operators in the field theory and define

$$\mathbf{T}\psi_a\mathbf{T}^\dagger = T_{ab}\psi_b, \quad \mathbf{C}\psi_a\mathbf{C}^\dagger = C_{ab}\psi_b^\dagger. \quad (15)$$

This and the T, C properties of \mathcal{H} (Eq. (1)) implies the invariance: $\mathbf{THT}^\dagger = H, \mathbf{CHC}^\dagger = H$.

Since right movers and left movers are $\psi_R \equiv \langle x|\sigma_x = +\rangle$ and left movers $\psi_L \equiv \langle x|\sigma_x = -\rangle$, the spinor field ψ has the block structure:

$$\psi = \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} \quad (16)$$

in the eigenbasis of σ_z . Upon passing to Euclidean space by $t \rightarrow -i\tau$, the Schrodinger equation for \mathcal{H} in Eq. (3), $i\partial_t\psi = \mathcal{H}\psi$, becomes $\partial_z\psi_R = \partial_{\bar{z}}\psi_L = 0$, where $\partial_{\bar{z}} = \partial_\tau + i\partial_x, \partial_z = \partial_\tau - i\partial_x$. This confirms the anticipated chirality of ψ_R and ψ_L .

The \mathbf{T} and \mathbf{P} transformations exchange left and right movers:

$$\begin{aligned} T: \quad \psi_R &\rightarrow -\eta_t\psi_L, \quad \psi_L \rightarrow \eta_t\psi_R \\ \tilde{T}: \quad \psi_R &\rightarrow \tilde{\eta}_t\psi_L, \quad \psi_L \rightarrow \tilde{\eta}_t\psi_R \end{aligned} \quad (17)$$

and

$$\begin{aligned} P: \quad \psi_R &\rightarrow \eta_p\psi_L, \quad \psi_L \rightarrow \eta_p\psi_R \\ \tilde{P}: \quad \psi_R &\rightarrow -\tilde{\eta}_p\psi_L, \quad \psi_L \rightarrow \tilde{\eta}_p\psi_R \end{aligned} \quad (18)$$

On the other hand, C transforms fields into their conjugates:

$$\begin{aligned} C: \quad \psi_R &\rightarrow \eta_c\psi_R^\dagger, \quad \psi_L \rightarrow -\eta_c\psi_L^\dagger \\ \tilde{C}: \quad \psi_R &\rightarrow \tilde{\eta}_c\psi_R^\dagger, \quad \psi_L \rightarrow \tilde{\eta}_c\psi_L^\dagger. \end{aligned} \quad (19)$$

Hence for the AZ classes A,C,D which do not have \mathbf{T} or \mathbf{P} symmetry, chiral states with only

ψ_R or ψ_L can be realized as edge states and are protected from a mass gap since mass term couples left and right.

We now use the \mathbf{T} symmetry to assign spins and check for spin-momentum locking. On physical grounds, we consider the smallest number of components in each class, i.e. either 1 or 2. It is well-known that \mathbf{T} has the representation $\mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on spin 1/2 particles and $\mathbf{T}^2 = -1$. Hence when the representation of \mathbf{T} involves $i\sigma_y$ or $i\tau_y$ and $\mathbf{T}^2 = -1$ in Table II, $\vec{\sigma}$ or $\vec{\tau}$ should act on the spin space. This is particularly interesting since $|\sigma_x = +\rangle$ and $|\sigma_x = -\rangle$ are right- and left-moving states by definition of the hamiltonian Eq. (3): this, as we mentioned earlier, is a manifestation of spin-momentum locking.

The classes with spin-momentum locking are $\mathbf{AII}_{(1)}$, $\mathbf{DIII}_{(1)}$, $\mathbf{CII}_{(1)}$. These are all TI-TS edge states of type \mathbb{Z}_2 within our scheme. For these, we can label the fields $\psi_R = \psi_{R\uparrow}$, $\psi_L = \psi_{L\downarrow}$. $\mathbf{AII}_{(1)}$ and $\mathbf{DIII}_{(1)}$ have well known examples. QSH edge states^{4,5,22} *in the absence of particle hole symmetry* are examples of $\mathbf{AII}_{(1)}$ class. Note that we derived here the spin-momentum locking, which arises from the spin-orbit coupling in QSH systems, on very general grounds. A 2d version of a He_3B superfluid phase where up-spin pairs and down-spin pairs have opposite angular momentum, would be an example of the $\mathbf{DIII}_{(1)}$ class.²³ Such a state has not been realized yet, but perhaps could be in a film geometry with control over the boundary conditions. $\mathbf{CII}_{(1)}$ can be realized²⁴ as a particle-hole doubled version of $\mathbf{AII}_{(1)}$ much the same way as how in 3d a \mathbf{CII} TI was constructed out of two copies of 3d Dirac Hamiltonian in Schnyder *et al.*⁹.

$\mathbf{DIII}_{(2)}$ and $\mathbf{CII}_{(2)}$ classes have both spin components for right-movers and left-movers each. The Kane-Mele (KM) model⁴ at zero chemical potential has particle-hole symmetry and hence does not strictly speaking belong to class \mathbf{AII} . Moreover the spin or charge edge current is absent as the current operators are odd under charge conjugation²⁵. Nevertheless, there is a charge neutral gapless edge mode^{25,26}. This is an example of $\mathbf{DIII}_{(2)}$ class²⁴. $\mathbf{CII}_{(2)}$ is unique

in that spin is tied to charge, i.e. particle-hole transformations flip spin: $(\psi_{R\uparrow}, \psi_{R\downarrow}) \rightarrow (-\psi_{R\downarrow}^\dagger, \psi_{R\uparrow}^\dagger)$. Note that these spin-momentum locking properties offer concrete distinctions between classes (DIII₍₁₎, CII₍₁₎) and (DIII₍₂₎, CII₍₂₎).

AIII₍₁₎, non-chiral **D**, and **BDI**₍₂₎ are spinless fermion. Note that we find the non-chiral **D** TI to be of \mathbb{Z}_2 type and distinct from the chiral **D** which is of \mathbb{Z} type. **BDI**₍₂₎ is of particular interest, since the zero field QHE in trigonally strained graphene^{27,28} is perhaps an example of a \mathbb{Z} TI this class²⁴.

VI. VARIATIONS OF LUTTINGER LIQUIDS

We are now in the position to consider how interactions consistent with the **T**, **C**, **P** symmetries could affect the $\bar{d} = 1$ edge states. In general, bulk interactions should lead to interactions on the edge. If the bulk stays gapped, one can focus on the edge states even in the presence of interactions. While the topological invariants based on single particle wave functions cannot be applied to interacting systems, the edge state theory can incorporate the effects of interactions.

The fractional quantum Hall effect (FQH) is the prime example. The FQH edge state resulting from Coulomb interaction in the bulk has no topological invariant associated with it, while the integer QHE is associated with the Chern number². However the fractional quantum Hall edge states are chiral Luttinger liquids which are related to the integer quantum Hall edge states (chiral Fermi liquid) by the addition of an exactly marginal perturbation to the Dirac action¹⁵. An exactly marginal perturbation on a non-interacting edge state preserves the gaplessness, but deforms it into an interacting theory with non-trivial exponents, fractional charges, etc.

Motivated by the FQH case, we classify the exactly marginal perturbations for each proposed TI-TS's in Table IV, as a way of characterizing the effect of bulk interactions.

The starting point is the action for the generic free Dirac Hamiltonian Eq. (14):

$$S = \int dx dt \left[\psi_R^\dagger (\partial_z + A_x + V_+) \psi_R + \psi_L^\dagger (\partial_{\bar{z}} - A_x + V_+) \psi_L + \left(\psi_L^\dagger (V_- + iA_y) \psi_R + h.c. \right) \right]. \quad (20)$$

Recall that ψ_R and ψ_L are vectors in the space represented by τ . V_+ can be interpreted as a chemical potential, or equivalently the time component of a gauge field as it couples to currents $\psi_R^\dagger V_+ \psi_R + \psi_L^\dagger V_+ \psi_L$. We set it to zero. If $V_- + iA_y$ is one dimensional, it simply corresponds to a complex mass. Hence removing A_y through a unitary transform U_θ is equivalent to removing the phase of the mass by redefining ψ_L . After removing A_y , and absorbing the physical gauge field A_x to the definition of the ψ fields, the action for the massless zero mode simplifies to

$$S = \int dx dt \left(\psi_R^\dagger \partial_z \psi_R + \psi_L^\dagger \partial_{\bar{z}} \psi_L \right). \quad (21)$$

We consider left-right current-current perturbations in analogy with Luttinger liquids and single out those preserving the **T**, **C**, **P** of the free theory. Consider the currents $J_L^a = \psi_L^\dagger t^a \psi_L$, $J_R^a = \psi_R^\dagger t^a \psi_R$, where t^a is a hermitian matrix acting on the τ space, and define the operator $\mathcal{O}^a = J_L^a J_R^a$ (no sum on a). Since ψ has scaling dimension 1/2, the operator \mathcal{O}^a has dimension two, i.e. it is marginal, and a term $g \mathcal{O}^a$ can be added to the lagrangian. For the $T, \tilde{T}, P, \tilde{P}$ symmetries, \mathcal{O}^a is invariant if the appropriate η commutes with t^a . For the C, \tilde{C} symmetries which transform fields into their conjugates, invariance of the operator additionally requires $(t^a)^T = \pm t^a$. The renormalization group beta function for \mathcal{O}^a is in general proportional to the quadratic Casimir for the Lie algebra generated by the t^a . If this beta function vanishes for a symmetry invariant \mathcal{O}^a , it is an exactly marginal perturbation.

For all TI-TS's, the marginal perturbation \mathcal{O}^a is invariant for $t^a = \mathbf{1}$, and we can consider the

action

$$S = \int dx dt \left(\psi_R^\dagger \partial_z \psi_R + \psi_L^\dagger \partial_{\bar{z}} \psi_L + g J_L J_R \right). \quad (22)$$

Since the currents $J_{L,R}$ are then $U(1)$ currents, the beta function vanishes making this perturbation exactly marginal. Eq. (22) describes different versions of Luttinger liquids for different classes.

The choice $t^a = \tau_y$, which requires at least 2 components for each chirality, also yields an invariant \mathcal{O}^a for the classes $\text{DIII}_{(2)}$ and $\text{CII}_{(1,2)}$. Since this involves a single t^a , it again generates a $U(1)$ current and the associated \mathcal{O}^a is again exactly marginal.

We list each exactly marginal perturbation for the above TI-TS's:

- **AII₍₁₎ and DIII₍₁₎**. Both are one-component spin-momentum locked classes. The only allowed perturbation is with $t^a = \mathbf{1}$:

$$\mathcal{O}^a = \left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right) \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} \right). \quad (23)$$

The so-called helical liquid for interacting QSH edge state²⁹ requires such a perturbation. Interestingly such a bulk interaction effect on the edge states has been recently confirmed^{25,26,30}.

- **DIII₍₂₎ and CII₍₂₎**. Both are two-component classes which can be perturbed with $t^a = \mathbf{1}$ and $t^a = \tau_y$. $t^a = \mathbf{1}$ yields the spin-full Luttinger liquid with

$$\mathcal{O}^a = \left(\psi_{L\uparrow}^\dagger \psi_{L\uparrow} + \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right) \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} + \psi_{R\downarrow}^\dagger \psi_{R\downarrow} \right). \quad (24)$$

Whereas $t^a = \tau_y$ turn J_L^a and J_R^a into a spin-singlet currents and

$$\mathcal{O}^a = - \left(\psi_{L\uparrow}^\dagger \psi_{L\downarrow} - \psi_{L\downarrow}^\dagger \psi_{L\uparrow} \right) \left(\psi_{R\uparrow}^\dagger \psi_{R\downarrow} - \psi_{R\downarrow}^\dagger \psi_{R\uparrow} \right). \quad (25)$$

These are new types of Luttinger liquids which we refer to as the “spin-singlet liquid”.

- **AIII₍₁₎, non-chiral D and BDI₍₂₎**. These are spinless fermion classes which can be

single component. They can only be perturbed with $t^a = \mathbf{1}$.

- **CII₍₁₎**. This has both particle and hole components with spin-momentum locking for each component. It is a different kind of Luttinger liquid, which we refer to as the “double helix”, since the free part is essentially a doubled KM model.

$$\mathcal{O}^a = \left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} + \psi_{L\downarrow}^{\prime\dagger} \psi_{L\downarrow}' \right) \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} + \psi_{R\uparrow}^{\prime\dagger} \psi_{R\uparrow}' \right) \quad (26)$$

Next consider adding more than one perturbation, i.e. $\sum_a g_a \mathcal{O}^a$. In general, the operator product expansion of \mathcal{O}^a with \mathcal{O}^b generates another \mathcal{O} operator associated with the current corresponding to $[t^a, t^b]$, and this gives rise to a renormalization group beta function proportional to the quadratic casimir of the Lie algebra generated by the t^a . Only classes $\text{DIII}_{(1)}$ and $\text{CII}_{(2)}$ have two allowed \mathcal{O}^a listed above: $t^a = \mathbf{1}$ or τ_y . However since these t^a commute, this two parameter perturbation is also exactly marginal. In summary, we find all possible symmetry preserving quartic interactions to be exactly marginal, deforming the free Dirac edge theory into an interacting one that preserves the gaplessness.

VII. CONCLUSIONS

We classified Dirac hamiltonians in one dimension according to the discrete symmetries of time-reversal, particle-hole and chiral symmetry, and found 17 inequivalent ones. Assuming that two-dimensional topological insulators (or superconductors) are realized on their one dimensional boundary as Dirac fermions, we found 11 of these classes that possessed a zero mode which was protected by the symmetries. This should be compared with the classifications based on bulk topological or boundary localization properties in^{9–11}, which predict 5 classes in any dimension. The classes we find beyond the standard 5 are in classes AIII, BDI, two versions of CII, a distinct version of DIII

and a \mathbb{Z}_2 version of D. We suggested that physical realizations for the new TI's in $\text{BDI}_{(2)}$ and $\text{CII}_{(1)}$ could perhaps be strained graphene and a doubled Kane-Mele model respectively.

The simplest interpretation of the existence of these new classes of TI in two spatial dimensions is that there are theories with boundary zero modes that are not necessarily protected by topology, and this is attributed to the richer structure of the classification of Dirac hamiltonians in 1 dimension. On the other hand, it remains a possibility that the new classes are characterized by some as yet unknown topological invariants.

We also studied possible manifestations of bulk interactions as quartic interactions on the boundary in two dimensions. For all classes of potential TI's, we found that all such interactions that preserve the discrete symmetries are exactly marginal. The exact marginality pre-

serves the gaplessness, but deforms the theory into distinct variations of Luttinger liquids.

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